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Beliefs and stochastic modelling of interest rate scenario risk

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Abstract. We present a framework that allows for a systematic assessment of risk given a specific model and belief on the market. Within this framework the time evolution of risk is modeled in a twofold way. On the one hand, risk is modeled by the time discrete and nonlinear garch(1,1) process, which allows for a (time–)local understanding of its level, together with a short term forecast. On the other hand, *via* a diffusion approximation, the time evolution of the probability density of risk is modeled by a Fokker-Planck equation. Then, as a final step, using Bayes theorem, beliefs are conditioned on the stationary probability density function as obtained from the Fokker-Planck equation. We believe this to be a highly rigorous framework to integrate subjective judgments of future market behavior and underlying models. In order to demonstrate the approach, we apply it to risk assessment of empirical interest rate scenario methodologies, *i.e.* the application of Principal Component Analysis to the the dynamics of bonds.

PACS. 05.45.Tp Time series analysis – 02.50.Ey Stochastic processes

1 Introduction

Risk or volatility behaviour of financial instruments may be described by nonlinear dynamics and the presence of noise, as it has been suggested in the finance [1] and econophysics [2] literature. The most important observation here is, that periods of high and low volatility tend to be separated. This usually is referred to as volatility clustering and might be observed in a number of different financial time series.

Distributions of solutions of such systems, which initially are Gaussian, will not remain so when the systems evolve. However, finance professionals like risk managers, heavily depend on modelling this behavior in order to quantify risk within their usually highly complex portfolios. In addition to that, portfolio managers have to express opinions on the market in order to gain from the future price behaviour of financial instruments in their portfolios. The key question then is to bridge the gap between a fine grained model of volatility, which allows for a local description of its current level together with short term forecasting and an understanding of the time evolution of its probability density function (pdf). The later to be an essential ingredient in order to understand the evolution of risk on longer time horizons and as a basis for an incooperation of beliefs.

In the present paper, we therefore address the problem of integrating beliefs into systems modeled by nonlinear stochastic differential equations. We present a framework where beliefs are considered to be drawn from Gaussian pdfs. Calculating a prior pdf from a Fokker-Planck equation (FPE), describing the time evolution of the volatility's pdf, we apply Bayes theorem in order to condition the beliefs on the prior knowledge. The resulting posterior distribution then generates scenarios where prior knowledge and beliefs are rigorously integrated.

Although the presented approach is general and might be applied to a wide range of asset classes, we demonstrate the approach on empirical scenario generation methodologies, namely the Principal Component Analysis (PCA) [3] of the yield curve dynamics [4,5].

2 Bond price dynamics

Consider daily price changes of Euro Government bonds from 7/1996 - 7/2000, as approximated by daily changes $\Delta p_{m,n} = p_{m,n} - p_{m,n-1}$ of REX price indices $p_{m,n}$, where $m = 1, ..., 10$ is the maturity and n is the discrete time in days. In order to construct an empirical model of the underlying dynamics we apply PCA to construct a linear orthogonal superposition of $i = 1, ..., N$ independent factors $s_{i,n}$ by an eigenvalue decomposition of the correlation matrix of the 10 time series $\Delta p_{m,n}$. The resulting model can be expressed by

$$\Delta p_{m,n} = \langle \Delta p_m \rangle_n + \sum_{i=1}^{N} w_{m,i} \Delta s_{i,n} , \qquad (1)$$

where $w_{m,i}$ are the factor loadings and $\langle ... \rangle_n$ expresses an average over n. In this model the total variance of $\Delta p_{m,n}$

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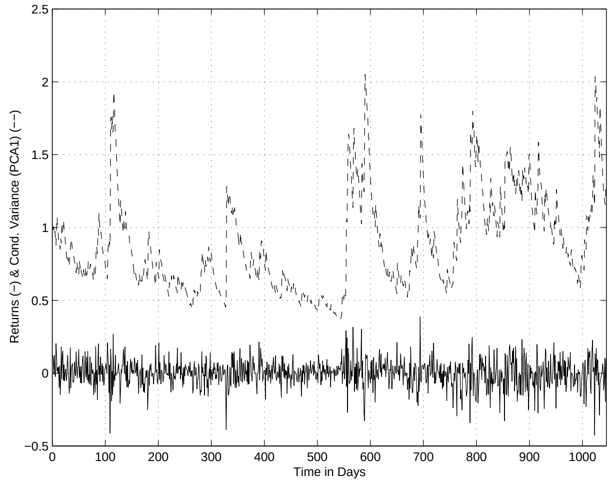

Fig. 1. The PCA generated factor $\Delta s_{1,n}$ (-), the shift, together with the garch(1,1) estimated conditional variance σ_n (- -) is shown. Volatility clustering is observable in both time series. For convenience has σ_n been multiplied by the factor 100.

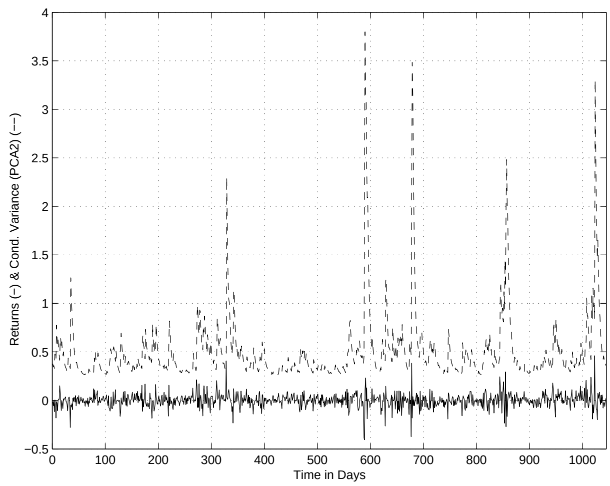

Fig. 2. The PCA generated factor $\Delta s_{2,n}$ (-), the twist, together with the garch(1,1) estimated conditional variance σ_n (- -) is shown. Volatility clustering is observable in both time series. For convenience has σ_n been multiplied by the factor 100.

is given by the sum of the variances of the single factors $s_{i,n}$. Furthermore does the average $\langle \Delta p_m \rangle_n$ become close to zero and might thus be neglected in the following.

The resulting first two factors $\Delta s_{1,n}$ and $\Delta s_{2,n}$ are shown in Figure 1 and Figure 2. Immediately one observes volatility clustering to be present in both time series, although regimes of high volatility tend to be more separated from regimes of low volatility in the series $\Delta s_{2,n}$. Since the first two factors show to explain more then 95 percent of the variance of the data, with the first and second component explaining about 80 and 15 percent respectively, a model containing only these factors might be considered as a good approximation to the dynamics. A closer examination of the factor loadings $w_{m,i}$ in Figure 3 allows us to interpret the dynamics in terms of a parallel shift over maturities, superimposed by a twist

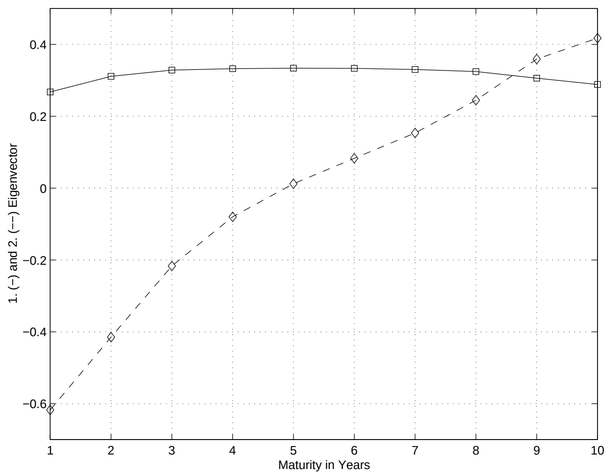

Fig. 3. The PCA generated factor loadings $w_{m,i}$ indicate that the price behavior of Euro Government bonds is dominated by a shift (-) along maturities, superimposed by a twist (- -) at 5 year maturity.

around $m = 5$. This type of behavior, of course, is known among finance professionals and exploited for dimension reduction in Monte-Carlo type interest rate scenario simulation and for structuring trades within bond portfolios.

3 Risk dynamics

3.1 Discrete-time garch modelling

Volatility clustering in the time series $\Delta s_{1,n}$ and $\Delta s_{2,n}$ and rather slow decaying autocorrelation functions of its squares $\Delta s_{1,n}^2$ and $\Delta s_{2,n}^2$, indicate that the time evolution of risk can be modeled by a discrete time garch(1,1) [6] process as defined for each factor separately by

$$\Delta s_n = \sigma_n Z_n \qquad (2)$$

$$\sigma_{n+1}^2 = \omega + \beta\sigma_n^2 + \alpha\sigma_n^2 Z_n^2 \,, \qquad (3)$$

where $\{Z_n\} \propto i.i.d.\ N(0,1)$ and σ_n^2 is termed conditional variance. Here and in the following we will drop the indices 1 and 2 whenever the equations are to be applied to both series, *i.e.* $\Delta s_{1,n}$ and $\Delta s_{2,n}$to become Δs_n and $\sigma_{1,n}$ and $\sigma_{2,n}$ to become σ_n. The process is linear in mean, nonlinear in variance and allows for an estimation of volatility in terms of conditional variance.

The volatility itself, in principle, is not observable, but has to be estimated by averaging over a specific time window or by derivation from other sources, like option implied volatilities. These empirical methods, however, suffer from the major drawback, that they either do lag behind the current volatility (through time averaging) or rely on financial instruments which possibly do not have enough market liquidity and therefore might result in a poor estimator. Equation (3) instead, allows for a direct estimation of risk from the underlying series and even enables for short term forecasting.

Performing a least squares fit on the data, one arrives at the following parameters for the processes as defined by equation (2) and equation (3), for the shift and twist respectively: $\omega = 0.0002$, $\beta = 0.0523$, $\alpha = 0.9238$ and $\omega = 0.0006, \beta = 0.1228, \alpha = 0.7613$. The estimation of the conditional variance σ_n^2 is shown in Figure 1 and Figure 2 together with the time series of $\Delta s_{1,n}$ and $\Delta s_{2,n}$. It can be observed that the conditional variances as estimated by the garch(1,1) process estimate the volatility of the factors on time and with good quality.

Unfortunately one is not able to derive a closed form expression for the stationary distribution of the garch process. However, such a distribution function together with an understanding of its time evolution is a crucial building block for risk management purposes.

3.2 Garch diffusion approximation

In order to obtain an expression for the stationary distribution of the garch(1,1) process we seek a diffusion approximation of the Markovian garch process by allowing the time intervals between successive time steps n to approach zero. Applying an approximation scheme of the form

$$\Delta s_{nh} = \sigma_{nh} Z_{nh} \tag{4}$$

$$\sigma^2_{(n+1)h} = \omega_h + \beta_h \sigma^2_{nh} + \frac{1}{h}\alpha_h \sigma^2_{nh} Z^2_{nh}\,, \tag{5}$$

with $\{Z_{nh}\} \propto i.i.d.\ N(0,h)$ and h being a discrete time interval, [7] obtains the following diffusion approximation in the limit $h \to 0$

$$\mathrm{d}s_t = \sigma_t \mathrm{d}w_{1,t} \tag{6}$$

$$\mathrm{d}\sigma_t^2 = (\omega + \Theta\sigma_t^2)\mathrm{d}t + \alpha\sigma_t^2 \mathrm{d}w_{2,t}\,, \tag{7}$$

together with a proof of existence of the stationary distribution. We find $\Theta < 0$, t to be the continuous time and $\mathrm{d}w_{1,t}$ and $\mathrm{d}w_{2,t}$ to be two independent standard Brownian motions, independent of initial values s_0 and σ_0^2. The terms $D_1(\sigma^2) = \omega + \Theta\sigma_t^2$ and $D_2(\sigma^2) = \alpha\sigma_t^2$ are determining the drift and diffusion of the process.

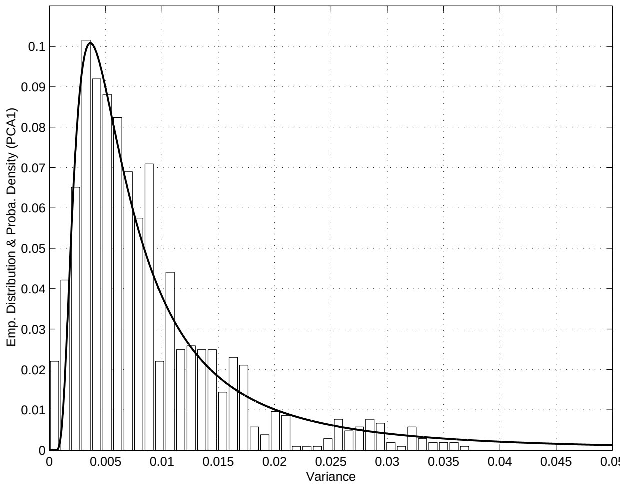

Fig. 4. The inverse gamma pdf for the shift is shown, together with the empirical distribution of the empirical variance.

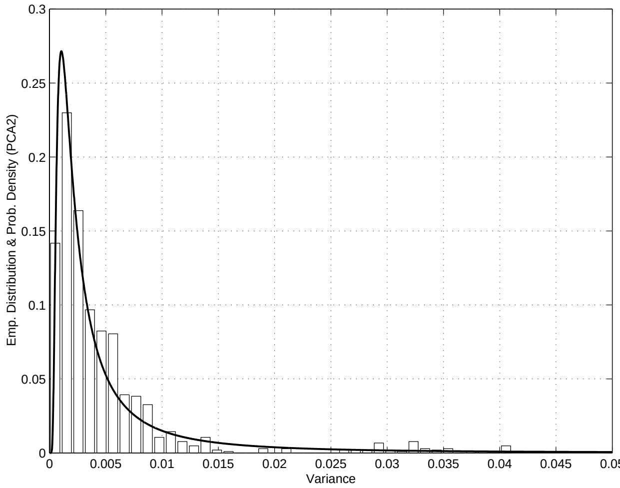

Fig. 5. The inverse gamma pdf for the twist is shown, together with the empirical distribution of the empirical variance.

4 Scenario generation

4.1 Fokker-Planck equation

The time evolution of the probability density $W(v,t)$ of finding the system, as described by the stochastic differential equation (Eq. (7)), in a state $v = \sigma^2$ after time t can be approximated be a Fokker-Planck or Forward-Kolmogorov equation [8]

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial v} D_1(v) W + \frac{\partial^2}{\partial v^2}\frac{D_2(v)^2}{2} W\,. \tag{8}$$

We like the reader to note that since the FPE is obtained from a higher order Kramers-Moyal expansion after truncation of terms higher then second order and since the sample paths of equation (7) are not continuous functions in time, the FPE yields only an approximate description of the time evolution. Furthermore it is silent about the time evolution of the higher moments.

Observing that in the stationary state the probability current as defined by equation (8) must vanish, one obtains the stationary distribution $W^\star(v)$ after integration of equation (8). Obtaining the stationary distribution of the FPE is not an easy task for general drift and diffusion terms and for higher dimensional processes it might only be obtained by numerical procedures. The present situation, however, is analytically tractable.

Inserting the drift and diffusion terms from equation (7) into equation (8) yields the Fokker-Planck equation

$$\frac{\partial W}{\partial t} = -\frac{\partial}{\partial v}(\omega + \Theta v) W + \frac{\partial^2}{\partial v^2}(\frac{v^2\alpha^2}{2}) W, \tag{9}$$

and one finds the conditional variance to be distributed as an inverse gamma distribution in the stationary state $W^\star(v)$

$$W^\star(v) = \frac{1}{b^a \Gamma(a)} v^{-(a+1)} \exp(-\frac{1}{bv}), \tag{10}$$

with parameters $a = 1 + 2\Theta/\alpha^2$ and $b = 2\omega/\alpha^2$. For illustration we show in Figure 4 and Figure 5 the empirical distribution function of the empirical variance of $\Delta s_{1,n}$ and

$\Delta s_{2,n}$ averaged over a time window of 5 days, together with an estimation of the stationary pdfs as given by equation (10). The results show to be in good agreement.

4.2 Bayes theorem

Up to now we have obtained a detailed model for the time evolution of risk. The time discrete formulation, on the one hand, allows for a fine grained model and short term forecasting. The diffusion approximation, on the other hand, together with the FPE, for an understanding of the time evolution of the distribution function towards its stationary state.

In order to integrate beliefs $\boldsymbol{x}$ into the modelling procedure we apply the Bayes theorem [9]

$$W(v|\boldsymbol{x}) = \frac{W(\boldsymbol{x}|v)W^{\star}(v)}{\int W(\boldsymbol{x}|v)W^{\star}(v)\mathrm{d}v}. \quad (11)$$

Here $W^{\star}(v)$ is the pdf as obtained form the FPE and contains all available information on risk regarding the modelling procedure. The pdf $W(\boldsymbol{x}|v)$ describes the belief on the future risk behavior, given the current level of risk. The resulting pdf $W(v|\boldsymbol{x})$, finally, allows for a systematic estimation of expected risk conditioned on the modelling procedure.

For the purpose of scenario generation we assume M beliefs x_j to be drawn from normal densities $N(o, v)$, resulting in the belief density

$$W(\boldsymbol{x}|v) = \prod_{j=1}^{M} \frac{1}{(2\pi v)^{1/2}} \exp(-\frac{1}{2v}(x_j - o)^2). \quad (12)$$

In order to satisfy the central limit theorem for the beliefs as implied in equation (12), it has to be ensured that a sufficient number of beliefs x_j are independently estimated and that no single belief dominates the fluctuations of the entire population of beliefs.

The posterior density, as obtained by combining equation (10) and equation (12), again yields an inverse gamma density, with new parameters $a_{\mathrm{new}} = a + n/2$ and $b_{\mathrm{new}} = \frac{b}{1+\frac{b}{2}\sum_{j=1}^{M}(x_j-o)^2}$.

4.3 Scenarios

Consider the prior pdfs $W^{\star}(v_1)$ and $W^{\star}(v_2)$ as shown in Figure 4 and Figure 5. Since the $\Delta s_{1,n}$ and $\Delta s_{2,n}$ are orthogonal and decorrelated for each n (as a result of PCA), the product $W^{\star}(v_{1,2}) = W^{\star}(v_1)W^{\star}(v_2)$ represents the prior information on total scenario risk.

Assume to have obtained beliefs for the future behavior of the risk of the shift v_1 and the risk of the twist v_2, *i.e.*

$$\boldsymbol{x}_1 = [0.01\,0.20\,0.50\,0.20\,0.12\,0.10\,0.20\,0.07\,0.02\,0.01]$$

and

$$\boldsymbol{x}_2 = [0.10\,0.20\,0.05\,0.20\,0.12\,0.10\,0.20\,0.10\,0.20\,0.12]\,,$$

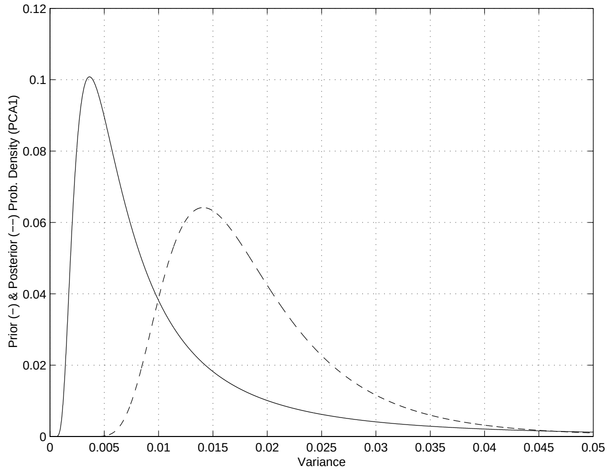

Fig. 6. The prior pdf for the shift (-) is shown, together with the posterior (- -) as obtained after integration of beliefs.

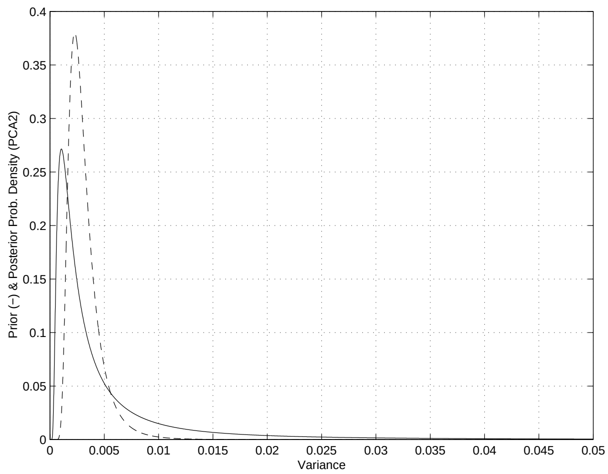

Fig. 7. The prior pdf for the twist (-) is shown, together with the posterior (- -) as obtained after integration of beliefs.

resulting from a variety of different sources like regression models, macroeconmic valuations or even opinions of experienced traders. The update of the prior, equation (10), with the beliefs, results in the inverse gamma posterior as shown in Figure 6 and Figure 7. The posterior $W(v_{1,2}|\boldsymbol{x}_{1,2}) = W(v_1|\boldsymbol{x}_1)W(v_2|\boldsymbol{x}_2)$ now contains all available information on risk for both, the shift and the twist. As it can be seen from the figures, the expectation regarding the risk of the shift is more pessimistic (resulting in a shift of the posterior along the x-axis as compared to the prior) and more heterogeneous (resulting in wider posterior as compared to the prior) then the expectation regarding the risk of the twist.

In a scenario generation context, this result would indicate that the portion of a trade regarding the twist will generate the desired profit more likely then the portion regarding the shift. A portfolio manager will therefore seek to hedge the portfolio exposure regarding the shift as good as possible in order exploit the expected dynamics of the twist.

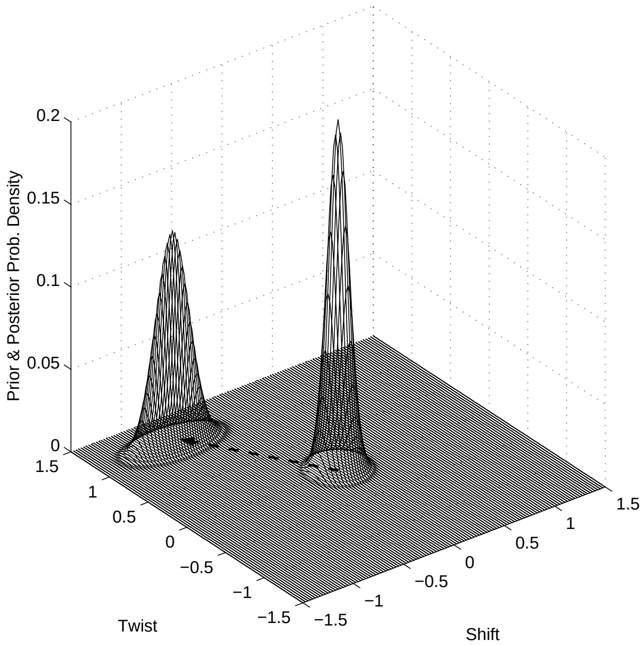

Fig. 8. Worst case scenarios in the shift-twist plane are shown. It is assumed that returns are normally distributed for time horizons much larger then 1 day. The prior worst case scenario is located at the origin, the current market state, whereas the posterior worst case scenario can be found at a location of the plane where the market is expected to be found after the time horizon under consideration.

An elegant visualization of this situation can be obtained by a plot in a shift-twist plane (Fig. 8). Recognizing that the returns of bonds (and most other financial instruments) are normally distributed on time horizons much larger then 1 day [2] one can estimate a scenario variance by specifying a given level of the probability density of volatility $W(v_{1,2}|\boldsymbol{x}_{1,2})$ as worst-case scenario, for example. In the figure a prior worst case scenario is shown at the origin of the plane, indicating the current state of the bond market, together with a corresponding posterior worst case scenario at a location of the plane where the portfolio manager expects the market to be found at the time horizon under his consideration.

5 Summary

We have presented a framework where beliefs on the market behaviour are integrated in a quantitative model of risk. Besides a (time–)local modelling of risk, *via* the garch process, a diffusion approximation is used to obtain a Fokker-Planck equation which describes the time evolution of the probability density function of risk towards its stationary state. Using Bayes theorem, subjective opinions on market behaviour are then integrated into the model.

While the framework does not allow for a faultless investment decision, it does help to condition (and visualize) forecasts on past experience and detailed models of the market. Doing so it allows for integrated risk management, whenever bets on the market are actively taken.

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